Then X, Y, Z are the mid-points of segments IA', IB', IC' respectively. Therefore XY is parallel to $A'B'$ and YZ is parallel to $B'C'$, so we have

$$\begin{aligned} \angle A'B'C' = \angle XYZ &= \tfrac{1}{2}\angle XIZ = \tfrac{1}{2}(180^\circ - \angle ZBX) \\ &= \tfrac{1}{2}(180^\circ - \angle ABC) = 90^\circ - \tfrac{1}{2}\angle ABC \end{aligned}$$

On the other hand, since BI bisects $\angle ABC$, we have

$$\angle C'BZ = \angle ZBI = \tfrac{1}{2}\angle ABC = \angle IBX = \angle XBA';$$

whence,

$$\begin{aligned} \angle C'BA' &= \angle C'BZ + \angle ZBI + \angle IBX + \angle XBA' \\ &= 4\cdot\tfrac{1}{2}\angle ABC = 2\angle ABC\,. \end{aligned}$$

Since A', B', C', and B are concyclic, we have $\angle C'BA' + \angle A'B'C' = 180^\circ$, which, on substitution, gives $2\cdot\angle ABC + \left(90^\circ - \tfrac{1}{2}\angle ABC\right) = 180^\circ$. Therefore, $\angle ABC = 60^\circ$.

3. For positive x_1, x_2, y_1, y_2, prove the inequality

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and Titu Zvonaru, Comănești, Romania. We give Alt's solutions and comment.

Solution 1. We have

$$\begin{aligned} (y_1+y_2)\left(\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2}\right) &= x_1^2 + x_2^2 + \frac{x_1^2 y_2}{y_1} + \frac{x_2^2 y_1}{y_2} \\ &\geq x_1^2 + x_2^2 + 2\sqrt{\frac{x_1^2 y_2}{y_1}\cdot\frac{x_2^2 y_1}{y_2}} \\ &= x_1^2 + x_2^2 + 2x_1x_2 = (x_1+x_2)^2\,. \end{aligned}$$

Solution 2. Let $a = \dfrac{x_1}{x_1+x_2}$ and $b = \dfrac{y_1}{y_1+y_2}$. Then $1-a = \dfrac{x_2}{x_1+x_2}$ and $1-b = \dfrac{y_2}{y_1+y_2}$, and the original inequality can be rewritten in the form

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \geq 1\,,$$

where a, $b \in (0,1)$. This is successively equivalent to

$$\begin{aligned} a^2(1-b) + b(1-a)^2 &\geq b(1-b)\,, \\ a^2 - a^2b + b - 2ab + a^2b &\geq b - b^2\,, \end{aligned}$$

or $a^2 + b^2 \geq 2ab$, which is true.

Solution 3. Since $\dfrac{x^2}{y} \ge 2x - y$ for $y > 0$, we apply this twice to obtain

$$\frac{a^2}{b} + \frac{(1-a)^2}{1-b} \;\ge\; (2a-b) + \big(2(1-a) - (1-b)\big) \;=\; 1\,,$$

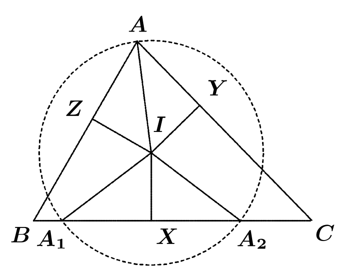
where a and b are defined as in Solution 2.

Comment. The original inequality is very simple relative to the high level of math olympiads, but it is a good occasion to perform different elementary techniques at an introductory level and, as well, for generalizations obtained by applying the Cauchy-Schwarz Inequality to $(\sqrt{y_1}, \sqrt{y_2}, \dots, \sqrt{y_n})$ and $\left(\dfrac{x_1}{\sqrt{y_1}}, \dfrac{x_2}{\sqrt{y_2}}, \dots, \dfrac{x_n}{\sqrt{y_n}}\right)$, where each x_i and y_i is a positive number:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \;\ge\; \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}\,.$$

5. Let I be the incentre of triangle ABC. Let points $A_1 \neq A_2$ lie on the line BC, points $B_1 \neq B_2$ lie on the line AC, and points $C_1 \neq C_2$ lie on the line AB so that $AI = A_1I = A_2I$, $BI = B_1I = B_2I$, $CI = C_1I = C_2I$. Prove that $A_1A_2 + B_1B_2 + C_1C_2 = P$, where P is the perimeter of $\triangle ABC$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the write-up of Amengual Covas.



Let X, Y, Z be the feet of the perpendiculars from I to the sides BC, CA, AB respectively. Then $IX = IY = IZ$. Right triangles IAZ, IA_1X, IA_2X, IYA are congruent, and since X is the mid-point of segment A_1A_2, we have

$$A_1A_2 \;=\; A_1X + XA_2 \;=\; AZ + YA\,.$$

Similarly, we obtain

$$\begin{aligned} B_1B_2 &= B_1Y + YB_2 = BX + ZB \\ \text{and} \quad C_1C_2 &= C_1Z + ZC_2 = CY + XC\,. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1A_2 + B_1B_2 + C_1C_2 &= (AZ + YA) + (BX + ZB) + (CY + XC) \\ &= (AZ + ZB) + (BX + XC) + (CY + YA) \\ &= AB + BC + CA \;=\; P\,. \end{aligned}$$